Pricing of Two-Asset Options under the Jump Diffusion Stock Price Process

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Abstract: This paper presents the pricing and valuation of two-asset options. Assuming an arbitrage-free market, a pricing model of two-asset option is established by assuming that the prices of both underlying assets follow the Poisson jump-diffusion process. A closed-form expression for the option price is derived using risk-neutral valuation.

1. Introduction

Among the various types of exotic options available in the market, rainbow option is one of the most interesting. A rainbow option has payoff that is contingent on the prices of multiple underlying assets. In this paper, the author presents the pricing of a rainbow option with two underlying assets, which is referred to as the two-asset option. Two-asset options are often used by investors to lock in a benefit from the "best performance" of two alternative financial instruments. For example, an investor may have an interest to invest in stock index A and B, while being uncertain on which stock index will yield higher returns in the future. Therefore, he may choose to purchase a two-asset option to ensure a profit from the best returns of the two indices on the maturity date.

Consider a market consisting of *n* risky assets with prices denoted by S_i, *i*= {1,2…n}. While it is appealing to assume that each price follows a Geometric Brownian Motion (which leads to the Black-Scholes framework), in reality, the arrival of critical information causes discontinuous shocks to stock prices (i.e. jumps). Based on such observation, Merton established a jump-diffusion model for European option pricing in [2]. Further development along this path is presented in [3], where it is assumed that the relative magnitude of the price jump is governed by the relative importance of the information causing the jump. Jump-inducing information is classified into several categories according to its relative importance. Merton's model is thus modified and the pricing formula of European stock options is obtained. This paper contributes to the literature by extending the results of [2] and [3] to the pricing of the two-asset option, which to the best of our knowledge has not been the subject of a published study.

We assume that both assets obey the Poisson jump-diffusion. The pricing equations of the two-asset option are derived by using the no-arbitrage principle under a risk-neutral measure, and the corresponding option pricing formula is given.

2. Model Specifications

Consider a financial market consisting of three assets $\left(B(t), S_1(t), S_2(t)\right)$. $B(t)$ is the price of a risk-less asset (bonds), and $S_i(t), i = \{1,2\}$ denote the prices of two risky assets (usually stocks) that can be continuously traded. Suppose they all follow the Poisson jump process, that is, the uncertainty of the process includes a "normal" diffusion and an "abnormal" jump.

$$\frac{dS_1(t)}{S_1(t)} = (\mu_1 - \lambda_1\kappa_1)dt + \sigma_1 dW_t^1 + U_1 dq_t^1 \tag{1}$$

$$\frac{dS_2(t)}{S_2(t)} = (\mu_2 - \lambda_2\kappa_2)dt + \sigma_2 dW_t^2 + U_2 dq_t^2 \tag{2}$$

where:

μ_i and σ_i are the expected returns and volatilities of stock *i*, $W_t^i, i = \{1,2\}$ denotes two standard Brownian Motions with correlation ρ, q_t^i are two independent Poisson processes of strength λ_i. Namely:

$$dq_t^i = \begin{cases} 1, & \text{The probability of occurrence is } \lambda_i dt \\ 0, & \text{The probability of occurrence is } 1-\lambda_i dt \end{cases}$$

Moreover

U_i is the jump size of stock price *i* when the first Poisson jump occurs

$\kappa_i = \varepsilon(U_i)$, where $\varepsilon(.)$ denotes the unconditional expectation.

Let $\mu_i, \sigma_i, \lambda_i, \kappa_i$ be constant parameters, the solutions of equation (1) and (2) are respectively obtained by Doléans-Dade exponential formula.

$$S_1(t) = S_1(0)\exp\left(\left(\mu_1 - \lambda_1\kappa_1 - \frac{1}{2}\sigma_1^2\right)dt + \sigma_1 W_t^1\right)\prod_{j=0}^{q_t^1}\left(1 + U_{1j}\right)$$

$$S_2(t) = S_2(0)\exp\left(\left(\mu_2 - \lambda_2\kappa_2 - \frac{1}{2}\sigma_2^2\right)dt + \sigma_2 W_t^2\right)\prod_{j=0}^{q_t^2}\left(1 + U_{2j}\right)$$

Here U_{ij} is the *i*-th jump-size of stock *j* (j= {1,2}), which are independent and identically distributed with initial conditions $U_{i,0} = 0,\ i = \{1,2\}$.

3. The Pricing Equation

Let $W(t) = F(S_1, S_2, t)$ be the time-t value of the option given the price of the two stocks. F is second-order differentiable and continuous with respect to S_1 and S_2, and first-order differentiable with respect to *t*. From (1) and (2), the option value cam be described by the following process:

$$\frac{dW(t)}{W(t)} = (\mu_W - \lambda_1\kappa_{1W} - \lambda_2\kappa_{2W})dt + \sigma_{1W}dW_t^1 + \sigma_{2W}dW_t^2 + U_{1W}dq_t^1 + U_{2W}dq_t^2 \tag{3}$$

where,

μ_W is the expected return of options.

σ_{1W} is the volatility of the option return in a period of no jump.

U_{iW} is the size of the i-th jump.

- $\kappa_{iW} = \varepsilon(U_{iW})$.

By Ito's Lemma, we have:

$$\begin{aligned}\mu_w = [&\frac{1}{2}\sigma_1^2 S_1^2 F_{11} + p\sigma_1\sigma_2 S_1 S_2 F_{12} + \frac{1}{2}\sigma_2^2 S_2^2 F_{22} + (\mu_1 - \lambda_1 k_1)S_1 F_1 + (\mu_2 - \lambda_2 k_2)S_2 F_2 + \\ &F_i + \lambda_1\varepsilon_1(F(S_1(1+U_1), S_2, t) - F(S_1, S_2, t)) + \\ &\lambda_2\varepsilon_2(F(S_1, S_2(1+U_2), t) - F(S_1, S_2, t))] / F(S_1, S_2, t),\end{aligned} \tag{4}$$

$$\sigma_{1W} = \frac{\sigma_1 S_1 F_1(S_1, S_2, t)}{F(S_1, S_2, t)} \tag{5}$$

$$\sigma_{2W} = \frac{\sigma_2 S_2 F_2(S_1, S_2, t)}{F(S_1, S_2, t)} \tag{6}$$

$$U_{1w} = \frac{F(S_1(1+U_1), S_2, t) - F(S_1, S_2, t)}{F(S_1, S_2, t)},$$

$$U_{2w} = \frac{F(S_1, S_2(1+U_1), t) - F(S_1, S_2, t)}{F(S_1, S_2, t)}$$

The subscript of F represents partial differential. $\varepsilon_i, i = \{1,2\}$ are the expectation operators U_i.

For a portfolio consisting of two stocks $S_1(t)$, $S_2(t)$, option W(t), and a risk-free asset $B(t)$, with weights respectively given by $w_i, i = \{1,2,3,4\}$, where

$$\sum_{i=1}^{4} w_i = 1.$$

Let $P(t)$ be the time-t value of this portfolio, which can be shown to follow

$$\frac{dP(t)}{P(t)} = (\mu_P - \lambda_1\kappa_{1P} - \lambda_2\kappa_{2P})dt + \sigma_{1P}dW_t^1 + \sigma_{2P}dW_t^2 + U_{1P}dq_t^1 + U_{2P}dq_t^2 \quad (7)$$

Where

μ_P is the expected return of the portfolio

σ_{iP} is the volatility of portfolio returns when no jump occurs,

U_{iP} is the portfolio value jump size when the i-th jump occurs.

- $\kappa_{iP} = \varepsilon(U_{iP})$

From (1), (2), (3), we have:

$$\mu_P = w_1\mu_1 + w_2\mu_2 + w_3\mu_W + (1 - w_1 - w_2 - w_3)r$$
$$\sigma_{1P} = w_1\sigma_1 + w_3\sigma_{1W}$$
$$\sigma_{2P} = w_2\sigma_2 + w_3\sigma_{2W}$$
$$U_{1P} = w_1U_1 + w_3\frac{F(S_1(1+X_1), S_2, t) - F(S_1, S_2, t)}{F(S_1, S_2, t)}$$
$$U_{2P} = w_2U_2 + w_3\frac{F(S_1, S_2(1+X_1), t) - F(S_1, S_2, t)}{F(S_1, S_2, t)}$$

Select weights $w_1 = w_1^*,\ w_2 = w_2^*,\ w_3 = w_3^*$ so that

$$\sigma_{1P}^* = w_1^*\sigma_1 + w_3^*\sigma_{1W} = 0$$
$$\sigma_{2P}^* = w_2^*\sigma_2 + w_3^*\sigma_{2W} = 0$$

The portfolio value thus follows:

$$\frac{dP^*(t)}{P^*(t)} = (\mu_P^* - \lambda_1\kappa_{1P}^* - \lambda_2\kappa_{2P}^*)dt + U_{1P}^*dq_t^1 + U_{2P}^*dq_t^2 \quad (8)$$

Assuming that the jump risk in non-systematic, that is, the jump process is not driven by the market factors, then the expected return of the portfolio is equal to the risk-free rate r. Therefore:

$$\begin{cases} \omega_1^*(\mu_1 - r) + \omega_2^*(\mu_2 - r) + \omega_3^*(\mu_3 - r) = 0, \\ \omega_1^*\sigma_1 + \omega_3^*\sigma_{1w} = 0, \\ \omega_2^*\sigma_2 + \omega_3^*\sigma_{2w} = 0. \end{cases}$$

Substitute (4), (5), (6) into the above system of equations, we observe that F satisfies the following differential equation:

$$\begin{aligned} &\frac{1}{2}\sigma_1^2S_1^2F_{11} + \rho\sigma_1\sigma_2S_1S_2F_{12} + \frac{1}{2}\sigma_2^2S_2^2F_{22} + (r - \lambda_1k_1)S_1F_1 + (r - \lambda_2k_2)S_2F_2 - rF + F_i + \\ &\lambda_1\varepsilon_1(F(S_1(1+U_1), S_2, t) - F(S_1, S_2, t) + \lambda_2\varepsilon_2(F(S_1, S_2(1+U_2), t) - F(S_1, S_2, t)) = 0. \end{aligned} \quad (9)$$

Note that if $\lambda_1 = \lambda_2 = 0$, then we get the pricing equation of a two-asset option whose underlying stock prices follow continuous processes (see [1]):

$$\frac{1}{2}\sigma_1^2 S_1^2 F_{11} + \rho\sigma_1\sigma_2 S_1 S_2 F_{12} + \frac{1}{2}\sigma_2^2 S_2^2 F_{22} + rS_1F_1 + rS_2F_2 - rF + F_i = 0.$$

4. Option Pricing Formula

For a European two-asset call option, F satisfies equation (9) with terminal condition:

$$F(S_1, S_2, T) = max(S_1, S_2) \tag{10}$$

T is the expiry of the option. This indicates an option payoff of:

$$\max(S_1(T), S_2(T)) = S_1(T) + \max(S_2(T) - S_1(T), 0).$$

Therefore, the two-asset European option can be decomposed into a combination of a risky asset and an asset exchange option.

For options with only one risky asset, let $= T - t$, the following holds:

It satisfies

$$\begin{cases} \frac{1}{2}\sigma_1^2 S_1^2 F_{11} + (r - \lambda_1 k_1)S_1F_1 - rF - F_\tau + \lambda_1\varepsilon(F(S_1(1+U_1),\tau) - F(S_1,\tau)) = 0, \\ F(S_1, 0) = S_1. \end{cases} \tag{11}$$

Theorem 1: The solution to equation (11) is given by:

$$F(S_1, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda_1\tau(\lambda_1\tau)^n}}{n!}\varepsilon_n(S_1\prod_{i=0}^{n}(1+U_{1i})e^{-\lambda_1 k_1\tau}). \tag{12}$$

Proof**:**

The partial derivatives for (12) are:

$$S_1F_1 = S_1\sum_{n=0}^{\infty}\frac{e^{-\lambda_1\tau(\lambda_1\tau)^n}}{n!}\varepsilon_n(\prod_{i=0}^{n}(1+U_{1i})e^{-\lambda_1\tau(\lambda_1\tau)}) = F,$$

$$S_1^2F_{11} = 0,$$

$$F_\tau = -\lambda_1 F + \lambda_1\sum_{n=1}^{\infty}\frac{e^{-\lambda_1\tau(\lambda_1\tau)^{(n-1)}}}{(n-1)!}\varepsilon_n(S_1\prod_{i=0}^{n}(1+U_{1i})\ e^{-\lambda_1 k_1\tau}) - \lambda_1 k_1 F.$$

Since

$$\varepsilon_{1+U_1}(F(S_1(1+U_1),\tau)) = \varepsilon_{1+U_1}(\sum_{n=0}^{\infty}\frac{e^{-\lambda_1\tau(\lambda_1\tau)^n}}{n!}\varepsilon_n(S_1\prod_{i=0}^{n}(1+U_{1i})e^{-\lambda_1 k_1\tau})$$

$$= \sum_{n=0}^{\infty}\frac{e^{-\lambda_1\tau(\lambda_1\tau)^n}}{n!}\varepsilon_{n+1}(S_1\prod_{i=0}^{n}(1+U_{1i})e^{-\lambda_1 k_1\tau}),$$

We have:

$$\frac{1}{2}\sigma_1^2 S_1^2 F_{11} + (r - \lambda_1 k_1) S_1 F_1 - rF - F_\tau =$$

$$(r - \lambda_1 k_1)F - rF + \lambda_1 F + \lambda_1 k_1 F - \lambda_1 \sum_{m=0}^{\infty} \frac{e^{-\lambda_1 \tau (\lambda_1 \tau)^m}}{m!} \varepsilon_{m+1}(S_1 \prod_{i=0}^{m+1}(1+U_{1i}) e^{-\lambda_1 k_1 \tau}) =$$

$$-\lambda_1 \varepsilon_{1+U_1}(F(S_1(1+U_1),\tau) - F(S_1,\tau)).$$

So $F(S_1,\tau)$ meets the terminal condition $F(S_1,\tau) = S_1$.

In addition, $\lim_{\tau \to 0} \varepsilon_n (S_1 \prod_{i=0}^{n}(1+U_{1i}) e^{-\lambda_1 k_1 \tau}) = \varepsilon_n (S_1 \prod_{i=0}^{n}(1+U_{1i})) = S_1(1+k_1))^n$, Therefore

$$\lim_{\tau \to 0} \sum_{m=1}^{\infty} \frac{e^{-\lambda_1 \tau (\lambda_1 \tau)^n}}{n!} \varepsilon_n (S_1 \prod_{i=0}^{n}(1+U_{1i}) e^{-\lambda_1 k_1 \tau}) = \lim_{\tau \to 0} \sum_{n=1}^{\infty} \frac{S_1 e^{-\lambda_1 \tau}(\lambda_1 \tau (1+k_1))^n}{n!} =$$

$$\lim_{\tau \to 0} S_1 e^{-\lambda_1 \tau}(e^{(1+k_1)\lambda_1 \tau} - 1) = 0.$$

So

$$\lim_{\tau \to 0} F(S_1,\tau) = \lim_{\tau \to 0} e^{-\lambda_1 \tau} \varepsilon_0 (S_1(1+U_{10}) e^{-\lambda_1 k_1 \tau}) = S_1.$$

Theorem 2: For asset exchange options, the following holds

$$\begin{cases} \frac{1}{2}\sigma_1^2 S_1^2 F_{11} + \rho \sigma_1 \sigma_2 S_1 S_2 F_{12} + \frac{1}{2}\sigma_2^2 S_2^2 F_{22} + (r - \lambda_1 k_1) S_1 F_1 + (r - \lambda_2 k_2) S_2 F_2 - rF - F_\tau + \\ \lambda_1 \varepsilon_1 (F(S_1(1+U_1), S_2, \tau) - F(S_1, S_2, \tau)) + \lambda_2 \varepsilon_2 (F(S_1, S_2(1+U_1), \tau) - F(S_1, S_2, \tau)) = 0, \\ F(S_1, S_2, 0) = (S_2 - S_1)^*. \end{cases}$$

The solution of the definite problem is

$$F(S_1,S_2,\tau) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{e^{-(\lambda_1+\lambda_2)(\lambda_1\tau)^m(\lambda_1\tau)^n}}{m!\,n!} \varepsilon_{mn}(V(S_1 \prod_{i=0}^{m}(1+U_{1i}) e^{-\lambda_1 k_1 \tau}, S_2 \prod_{i=0}^{n}(1+U_{2i}) e^{-\lambda_2 k_2 \tau}, \tau)). \quad (13)$$

Proof: See [5].

In summary, by superposition principle as well as the results of (12) and (13), the following results are obtained.

Theorem 3: Under the Poisson Jump-Diffusion stock price model given by equation (1) and (2), the risk-neutral value of a two-asset option with payoff in equation (10) is given by:

$$F(S_1,S_2;t,T) = \sum_{n=0}^{\infty} \frac{e^{-\lambda_1(T-t)(\lambda_1(T-t))^n}}{n!} \varepsilon_n (S_1 \prod_{i=0}^{n}(1+U_{1i}) e^{-\lambda_1 k_1 (T-t)}$$

$$+ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{e^{-(\lambda_1+\lambda_2)(\lambda_1(T-t))^m(\lambda_2(T-t))^n}}{m!\,n!}$$

$$\varepsilon_{mn}(V(S_1 \prod_{i=0}^{m}(1+U_{1i}) e^{-\lambda_1 k_1 (T-t)}, S_2 \prod_{i=0}^{n}(1+U_{2i}) e^{-\lambda_2 k_2 (T-t)}, (T-t))).$$

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